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PRELIMINARY INVESTIGATION OF
A CALCULUS OF FUNCTIONAL DIFFERENCES:
FIXED DIFFERENCES

Bruce J. MacLennan

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Rear Admiral R. H. Shumaker
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We introduce a notion of functional differences in which the difference of a function f with respect to a function h is that function g that describes how the value of f changes when its argument is altered by h: $f(h\ x) = g(f\ x)$. We also introduce the inverse operation of functional integration and derive useful properties of both operations. The result is a calculus that facilitates derivation and reasoning about recursive programs. This is illustrated in a number of simple examples. The present report presents preliminary results pertaining to fixed differences, that is, functional differences that do not

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depend on the value of the argument x.

Preliminary Investigation Of A Calculus of Functional Differences: Fixed Differences

B. J. MacLennan
Computer Science Department
Naval Postgraduate School
Monterey, CA 93943

Abstract

We introduce a notion of functional differences in which the difference of a function f with respect to a function h is that function g that describes how the value of f changes when its argument is altered by h: $f(h\ x) = g(f\ x)$. We also introduce the inverse operation of functional integration and derive useful properties of both operations. The result is a calculus that facilitates derivation and reasoning about recursive programs. This is illustrated in a number of simple examples. The present report presents preliminary results pertaining to *fixed differences,* that is, functional differences that do not depend on the value of the argument x.

1. Motivation

Simple recursive definitions often take the following form:

$$f\ x_0 = y_0$$

$$f\ (h\ x) = g\ (f\ x),\ \text{for}\ x \neq x_0$$

The assumption here is that an arbitrary domain value x can be reached by finitely many applications of h. That is, for all acceptable x there is an n such that $x = h^n\ x_0$. More general patterns of recursive definition will be considered later.

In deriving a recursive definition for a particular f, there are four unknowns that must be found, g, h, x_0 and y_0. Since h and x_0 are usually determined by the domain in question (e.g., they are zero and the successor function for the domain of natural numbers), and y_0 is usually easily determined from the definition of f, the main problem is determining the function g.

To see how this can be done consider the second equation above:

$$f\ (h\ x) = g\ (f\ x)$$

The function g tells us how much the value of the function f changes when its argument is changed by h. That is, if f's argument is changed by h, then its value is changed by g. This equation is analogous to the

finite difference equation

$$f\ (h + x) = g + (f\ x)$$

The difference is that in the first equation the "amounts of change" are expressed as functions rather than numbers, as they are in the finite difference equation. This is because we want to be able to deal with functions whose domains and ranges are nonnumeric (e.g., lists, sets, relations).

Based on this analogy we introduce

Definition 1: We define g to be the *functional difference* of f with respect to h if and only if it satisfies

$$f\ (h\ x) = g\ (f\ x).$$

We also call g "the change in f with respect to h" or "the change in f given the change h."

The next several sections will investigate the properties of functional differences[1].

The above equation defines the functional difference implicitly; to get an explicit definition we need to solve it for g. Therefore, eliminate x by use of the composition operation:

$$f \circ h = g \circ f$$

This can be solved by composing the inverse of f, f^{-1}, on both sides of the equation to yield:

$$g = f \circ h \circ f^{-1}$$

This seems to yield an explicit formula for the functional difference, but it is necessary to be more careful. First, f^{-1} will not be a function unless f is an isomorphism (one-to-one). Therefore we will assume this for the time being. Second, $f \circ f^{-1}$ is not the total identity function; rather, it is $\mathbf{I}_{\mathrm{rng}\, f}$, the identity function restricted to the range of f. Hence, doing the derivation more carefully, we have:

$$g \circ f = f \circ h$$

$$g \circ f \circ f^{-1} = f \circ h \circ f^{-1}$$

$$g \circ \mathbf{I}_{\mathrm{rng}\, f} = f \circ h \circ f^{-1}$$

Therefore, any solution to the difference equation, when restricted to rng f, will be $f \circ h \circ f^{-1}$. This is

1. A different notion of functional differences is described in [Paige&Koenig82].

summarized in

Theorem 1: Let f be a one-to-one function. If there is a relation g satisfying the equation $f \circ h = g \circ f$, then $g \circ \mathbf{I}_{\mathrm{rng}\, f} = f \circ h \circ f^{-1}$

Proof: Presented above. □

Thus $f \circ h \circ f^{-1}$ is a subfunction of every solution to the difference equation. Note that this theorem does *not* say that $f \circ h \circ f^{-1}$ is itself a solution. This result is proved in

Theorem 2: Let f be a one-to-one function. If the functional equation $f \circ h = g \circ f$ has a solution, then $f \circ h \circ f^{-1}$ is a solution.

Proof: For convenience we represent composition by juxtaposition when no ambiguity will result. By hypothesis the difference equation has a solution, so let a solution g be chosen; we must show that

$$fh = (fhf^{-1})f$$

We simplify the right hand side as follows:

$$(fhf^{-1})f = fh(f^{-1}f) = fh\mathbf{I}_{\mathrm{dom}\, f}$$

Since $fh = gf$ we know $\mathrm{dom}\, fh = \mathrm{dom}\, gf$. But $\mathrm{dom}\, gf \subseteq \mathrm{dom}\, f$ for all compositions, so $\mathrm{dom}\, fh \subseteq \mathrm{dom}\, f$. From this it follows that restricting the domain of fh to $\mathrm{dom}\, f$ is in fact no restriction, so we have $fh\mathbf{I}_{\mathrm{dom}\, f} = fh$ and the corollary is proved. □

This result permits us to call fhf^{-1} the *minimum* solution of the difference equation. In the future, when we speak of *the* functional difference of f with respect to h, we will mean the minimum solution g of $fh = gf$, which is fhf^{-1}.

Note however that fhf^{-1} being a solution is contingent upon the existence of *some* solution to the equation. The conditions for a solution existing are stated in

Theorem 3: Let f be a one-to-one function. The difference equation $f \circ h = g \circ f$ has a solution if and only if $\mathrm{dom}\, (f \circ h) \subseteq \mathrm{dom}\, f$.

Proof: To show the "if" part we assume $\mathrm{dom}\, fh \subseteq \mathrm{dom}\, f$. Then, substituting fhf^{-1} for g in the difference equation we have (as in the proof of Thm. 2):

$$(fhf^{-1})f \;=\; fh(f^{-1}f) \;=\; fh\mathbf{I}_{\mathrm{dom}\, f} \;=\; fh$$

The rightmost equality follows from the assumption.

To show the "only if" part we assume that $fh = gf$ has a solution. Then, as in the proof of Cor. 1-1, we have

$$\mathrm{dom}\; fh \;=\; \mathrm{dom}\; gf \;\subseteq\; \mathrm{dom}\; f$$

Hence $\mathrm{dom}\; fh \subseteq \mathrm{dom}\; f$. □

We observe in passing that none of the preceding results depend on h being one-to-one, or even a function. Indeed, they apply to any relation h. This leads us to investigate, in the next section, the isomorphic images of relations.

2. Isomorphic Images

In [MacLennan83] we define the *isomorphic image* of a relation R under a function f by

$$f \,\$\, R \;=\; \{ <fx,\, fy> \mid <x,\, y> \in R \}$$

This can be read "the f isomorphism of R" or "the isomorphic image under f of R." Note however that $f \,\$\, R$ is isomorphic to R only if f is defined for all members of R, otherwise $f \,\$\, R$ is isomorphic to a subrelation of R. Hence we introduce:

Definition 2: The members of a relation is the union of its domain and range:

$$\mathrm{mem}\; R \;=\; \mathrm{dom}\; R \cup \mathrm{rng}\; R$$

Definition 3: The isomorphism f is *defined* on R if and only if $\mathrm{mem}\; R \subseteq \mathrm{dom}\; f$.

If f is treated as a relation, and composition is allowed between relations, then we can derive an expression for the isomorphic image in terms of composition:

$$
\begin{aligned}
&<u,\, v> \in (f \,\$\, R) \\
&\iff \exists\, x,\, y\; [<x,\, y> \in R \wedge u = fx \wedge v = fy] \\
&\iff \exists\, x,\, y\; [<x,\, y> \in R \wedge <x,\, u> \in f \wedge <y,\, v> \in f] \\
&\iff \exists\, x,\, y\; [<u,\, x> \in f^{-1} \wedge <x,\, y> \in R \wedge <y,\, v> \in f] \\
&\iff \exists\, y\; [<u,\, y> \in (R \circ f^{-1}) \wedge <y,\, v> \in f]
\end{aligned}
$$

$\iff \langle u, v \rangle \in (f \circ R \circ f^{-1})$

Hence we have

$$f \,\$\, R = f \circ R \circ f^{-1}$$

This is of course exactly our formula for the functional difference. Note however that $f \,\$\, h$ is defined for all f and h, but that it is a solution to $fh = gf$ only if that equation has a solution. This is summarized in

Theorem 4: Let f be an isomorphism. Then

$$f \circ h = (f \,\$\, h) \circ f$$

if and only if dom $(f \cdot h) \subseteq$ dom f.

Proof: This follows from Thm. 2. □

Corollary 4-1: If f is one-to-one and dom $h \subseteq$ dom f, then $f \,\$\, h$ is the functional difference of f with respect to h.

Proof: Since dom $fh \subseteq$ dom $h \subseteq$ dom f we can apply the preceding theorem. □

Corollary 4-2: If f is one-to-one and defined on h, then $f \,\$\, h$ is functional difference of f with respect to h.

Proof: Since f is defined for h, mem $h \subseteq$ dom f. But dom $h \subseteq$ mem h, so dom $h \subseteq$ dom f and the previous corollary applies. □

Corollary 4-3: The functional difference of f with respect to h, if it exists, is $f \,\$\, h$, the f isomorphism of h.

Notice again that these results apply to *any* relation h; thus we will be able to take functional differences with respect to any relations (i.e., regardless of whether they are one-to-one or even functions). We will exploit this generality in the later development of the calculus. On the other hand, we still require f to be one-to-one.

Because isomorphism satisfies the difference equation (if anything does) we can read '$f \$ h$' either as "the isomorphic image under f of h," or as "the functional difference of f with respect to h." We call $f \$ h$ the functional difference regardless of whether the existence condition, dom $fh \subseteq$ dom f, is satisfied. That is, the functional difference exists, even if there is no functional difference equation that it satisfies.

3. Hasse Diagrams

We will exploit the relation between isomorphic images and functional differences so as to better understand the latter. In particular we can learn a lot about functional differences by looking at their *Hasse diagrams*.

The Hasse diagram of a relation R is constructed as follows. The diagram is a directed graph that has a vertex for every member of R (i.e., for every domain or range element of R). An edge goes from vertex x to vertex y if and only if $<x, y> \in R$.

If R is a function then there is at most one edge leading *out of* each vertex in its Hasse diagram. Further, if R is an isomorphism, then there is also at most one edge leading *into* each vertex. We will be most interested in a restricted class of isomorphisms called *sequences,* which are connected one-to-one functions. The Hasse diagram for a finite sequence has the form:

$$x_0 \to x_1 \to x_2 \to \quad \cdots \quad \to x_{n-1} \to x_n$$

We will often write such a relation in an abbreviated form:

$$(x_0, x_1, x_2, \ldots, x_{n-1}, x_n)$$

If R is an infinite sequence, then its Hasse graph must be a sequence that is infinite on either or both ends. We will be most interested in the case in which the sequence is *well founded,* that is, has an initial member:

$$x_0 \to x_1 \to x_2 \to \quad \cdots$$

x_0 is called the *initial member* [Carnap58, MacLennan83] of the sequence. It will also be convenient to write these relations in an abbreviated form:

$$(x_0, x_1, x_2, \cdots)$$

This case is of most interest to us, for it represents a function h such that any x_i can be reached from x_0 by

a finite number of applications of h. In particular

$$x_i = h^i x_0$$

This is exactly the situation we postulated in our discussion of functional differences.

Next we consider the Hasse diagrams of functional differences. Therefore, suppose that h is a finite or infinite sequence and that x_0 is the initial member of h. This relation can be diagramed:

$$x_0 \xrightarrow{h} x_1 \xrightarrow{h} x_2 \xrightarrow{h} x_3 \xrightarrow{h} \cdots$$

For each i define $y_i = fx_i$. We can diagram f:

$$\begin{tikzcd}
x_0 \arrow[d] & x_1 \arrow[d] & x_2 \arrow[d] & x_3 \arrow[d] & \cdots \\
y_0 & y_1 & y_2 & y_3 & \cdots
\end{tikzcd}$$

By applying f to each member of h we get the isomorphic image $f \,\$\, h$:

$$\begin{tikzcd}
y_0 \arrow[r] & y_1 \arrow[r] & y_2 \arrow[r] & y_3 \arrow[r] & \cdots
\end{tikzcd}$$

It is easy to see that this relation represents the functional difference of f with respect to h. Call the diagramed relation g and observe $y_{i+1} = g\ y_i$. Then,

$$f\ (h\ x_i) = fx_{i+1} = y_{i+1} = g\ y_i = g\ (fx_i)$$

Thus $f \circ h = g \circ f$.

Combining the diagrams for f, g and h we have:

$$\begin{tikzcd}
h = & x_0 \arrow[r] \arrow[dd] & x_1 \arrow[r] \arrow[dd] & x_2 \arrow[r] \arrow[dd] & x_3 \arrow[r] \arrow[dd] & \cdots \\
f = & & & & & \\
g = & y_0 \arrow[r] & y_1 \arrow[r] & y_2 \arrow[r] & y_3 \arrow[r] & \cdots
\end{tikzcd}$$

This diagram makes it apparent that $g = f \circ h \circ f^{-1}$, since to get from y_i to y_{i+1} we go backwards along an f arrow, forward on an h arrow, and forward on an f arrow. Notice however that if f is not defined for

some x_i then g will be isomorphic to a part rather than all of h. In fact g is isomorphic to all of f only if f is defined for all x_i (i.e., mem $h \subseteq$ dom f).

4. Properties of Differences

We develop a number of simple, useful properties of functional differences.

Theorem 5: The functional difference of the total identity with respect to any function is that function:

$$\mathbf{I} \,\$\, h \;=\; h$$

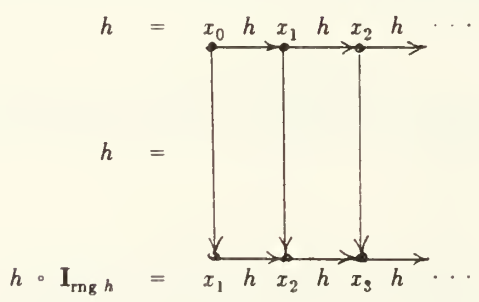
Proof: Derive $\mathbf{I} \,\$\, h \;=\; \mathbf{I} \circ h \circ \mathbf{I}^{-1} \;=\; h \circ \mathbf{I} \;=\; h.$ □

Theorem 6: The functional difference of a function with respect to itself is itself (but restricted to its range):

$$h \,\$\, h \;=\; h \circ \mathbf{I}_{\mathrm{rng}\, h}$$

Proof: We derive: $h \,\$\, h \;=\; h \circ h \circ h^{-1} \;=\; h \circ \mathbf{I}_{\mathrm{rng}\, h}.$ □

This result is easily understood from the diagram:

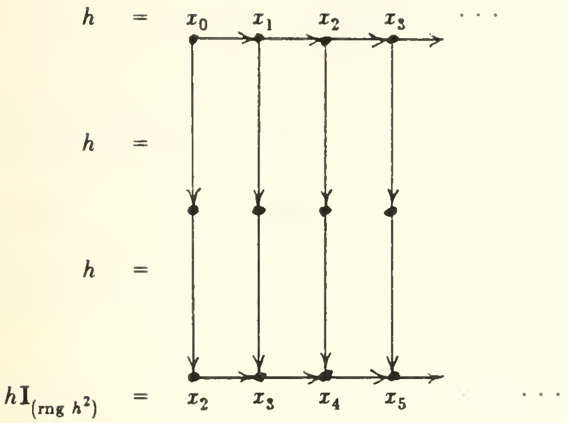


Corollary 6-1: The difference of a power of a function with respect to that function is that function (restricted to the range of the power):

$$h^n \,\$\, h \;=\; h \circ \mathbf{I}_{(\mathrm{rng}\, h^n)}$$

Proof: Derive $h^n \,\$\, h \;=\; h^n h h^{-n} \;\; h h^n h^{-n} \;=\; h\mathbf{I}_{(\,\mathrm{rng}\, h^n)}.$ □

For the case $n = 2$ this result is made clear by the diagram:



Theorem 7: The inverse of the difference of one function with respect to a second is the difference of the first with respect to the inverse of the second:

$$(f \$ h)^{-1} = f \$ h^{-1}$$

Proof: For the equality result we derive:

$$(f \$ h)^{-1} = (fhf^{-1})^{-1} = (f^{-1})^{-1}h^{-1}f^{-1} = fh^{-1}f^{-1} = f \$ h^{-1}$$

□

The theorem is obvious if we consider differences as isomorphic images: the inverse of the isomorphic image of a relation is the isomorphic image of the inverse relation.

Corollary 7-1: If mem $h \subseteq$ dom f, then $(f \$ h)^{-1} = f \$ h^{-1}$.

Proof: Under the given condition,

$$\text{dom } f(h \cup h^{-1}) \subseteq \text{dom } (h \cup h^{-1}) = \text{mem } h \subseteq \text{dom } f$$

Alternately observe that the existence of the two differences follows from these inequalities:

dom $fh \subseteq$ dom $h \subseteq$ mem $h \subseteq$ dom f

dom $fh^{-1} \subseteq$ dom h^{-1} = rng $h \subseteq$ mem $h \subseteq$ dom f

□

Theorem 8: The difference with respect to a composition of functions is the composition of the differences with respect to each of the functions:

$$f \mathbin{\$} (g \circ h) = (f \mathbin{\$} g) \circ (f \mathbin{\$} h)$$

provided dom $g \subseteq$ dom f or rng $h \subseteq$ dom f.

Proof: We begin with the right-hand side:

$$\begin{aligned}
(f \mathbin{\$} g)(f \mathbin{\$} h) &= (fgf^{-1})(fhf^{-1}) \\
&= fg(f^{-1}f)hf^{-1} \\
&= f(g\mathbf{I}_{\text{dom } f}h)f^{-1} \\
&= f \mathbin{\$} (g\mathbf{I}_{\text{dom } f}h)
\end{aligned}$$

Now, if dom $g \subseteq$ dom f then $g\mathbf{I}_{\text{dom } f} = g$, whereas if rng $h \subseteq$ dom f then $\mathbf{I}_{\text{dom } f}h = h$. So in either case $g\mathbf{I}_{\text{dom } f}h = gh$ and we have $(f \mathbin{\$} g)(f \mathbin{\$} h)$ is $f \mathbin{\$} gh$. □

This theorem provides a kind of chain rule for evaluating differences.

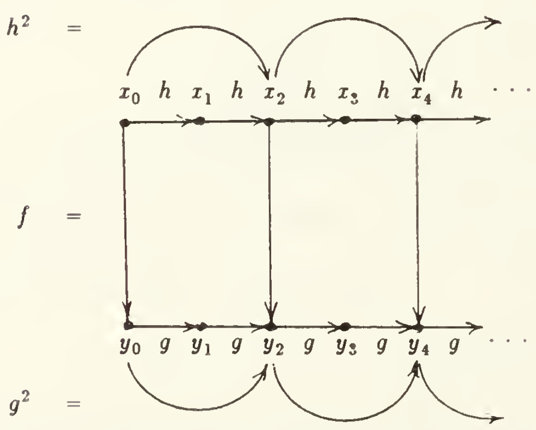
Corollary 8-1: The difference with respect to the nth power of a function is the nth power of the difference:

$$f \mathbin{\$} h^n = (f \mathbin{\$} h)^n$$

provided dom $h \subseteq$ dom f or rng $h \subseteq$ dom f.

Proof: This is an inductive application of the previous theorem. □

The case where $n = 2$ is obvious from the diagram:



We use a product notation for compositions:

$$\prod_{i=1}^{n} F_i = F_1 \circ F_2 \circ \cdots \circ F_{n-1} \circ F_n$$

Using this notation we can express

Corollary 8-2: The difference with respect to a product of functions is the product of the differences with respect to each of those functions:

$$f \,\$\, \left(\prod_{i=1}^{n} h_i \right) = \prod_{i=1}^{n} (f \,\$\, h_i)$$

provided either the domain or range of each of those functions is a subset of the domain of f.

Proof: This is just an inductive application of the theorem. □

Corollary 8-3: If $x_n = h^n x_0$, then

$$f x_n = (f \,\$\, h)^n (f x_0)$$

That is, if $y_i = f x_i$ then $y_n = (f \,\$\, h)^n y_0$.

Proof: This is an induction based on

$$y_n = f\,(h\, x_{n-1}) = (f \,\$\, h)\,(f x_{n-1}) = (f \,\$\, h)\, y_{n-1}$$

□

Theorem 9: If $x_n = \left(\prod_{i=1}^{n} H_i \right) x_0$ and $y_i = F\, x_i$ then $y_n = \left(\prod_{i=1}^{n} F \,\$\, H_i \right) y_0$, provided that dom $(F \circ H_i) \subseteq$ dom F, for $1 \leqslant i \leqslant n$.

Proof: The notation implies that $H_1 H_2 \cdots H_n$ is a function. Expanding the product we have:

$$F\, x_n = F \left[\left(\prod_{i=1}^{n} H_i \right) x_0 \right] = \left(F \prod_{i=1}^{n} H_i \right) x_0 = (F H_1 H_2 \cdots H_{n-1} H_n)\, x_0$$

We know from the definition of functional difference that $FH_i = (F \,\$\, H_i) F$ so we can push the F to the right through the H_i:

$$
\begin{aligned}
FH_1H_2 \cdots H_{n-1}H_n &= (F \$ H_1)FH_2 \cdots H_{n-1}H_n \\
&= (F \$ H_1)(F \$ H_2)F \cdots H_{n-1}H_n \\
&\vdots \\
&= (F \$ H_1)(F \$ H_2) \cdots (F \$ H_n)F
\end{aligned}
$$

Thus we have

$$
F \circ \prod_{i=1}^{n} H_i = \left(\prod_{i=1}^{n} F \$ H_i \right) \circ F
$$

Hence,

$$
y_n = F\ x_n = \left(\prod_{i=1}^{n} F \$ H_i \right) (F\ x_0) = \left(\prod_{i=1}^{n} F \$ H_i \right) y_0
$$

which proves the theorem. □

This theorem tells us how to use functional differences to get from fx_0 to fx_n, provided x_n is reachable from x_0. It is a functional difference analogue of Taylor's Theorem.

Let '$[f \$]$' denote a *presection* [Wile73] of the isomorphism operator; that is,

$$
[f \$]\ h = f \$ h
$$

Since $[f \$]$ leaves h unspecified, we call $[f \$]$ the *indefinite functional difference* of f.

Theorem 10: The indefinite difference of the composition is the composition of the indefinite differences:

$$
[(f \circ g) \$] = [f \$] \circ [g \$]
$$

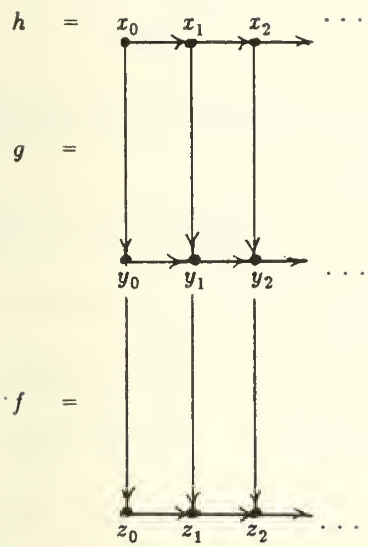
That is, $(f \circ g) \$ h = f \$ (g \$ h)$.

Proof: The derivation is direct:

$$
\begin{aligned}
fg \$ h &= (fg)h(fg)^{-1} \\
&= fghg^{-1}f^{-1} \\
&= f(ghg^{-1})f^{-1} \\
&= f(g \$ h)f^{-1} \\
&= f \$ (g \$ h)
\end{aligned}
$$

□

The theorem is obvious from its diagram:



The preceding theorem is a kind of chain rule for functional differences. It leads to

Corollary 10-1: The indefinite difference of the nth power is the nth power of the indefinite difference:

$$[f^n \ \$] = [f \ \$]^n$$

That is, $f^n \ \$ \ h = [f \ \$]^n h$.

Proof: This is just the inductive extension of the previous theorem. □

Corollary 10-2: The indefinite difference of the product is the product of the indefinite differences:

$$\left[\left(\prod_{i=1}^{n} f_i\right) \ \$\right] = \prod_{i=1}^{n} [f_i \ \$]$$

That is, $\left(\prod_{i=1}^{n} f_i\right) \ \$ \ h = \left(\prod_{i=1}^{n} [f_i \ \$]\right) \circ h$

Proof: This also follows inductively from the theorem. □

5. Examples of Differences

In this section we give several examples of functional differences. We begin with numerical functions, since they are most familiar. Let σ be the successor function:

$$\sigma = (0, 1, 2, \cdots)$$

Using the presection and postsection notations ($[a +]\ x = a + x$, $[-\ b]\ x = x - b$, etc.) we have the following differences (where we let '$\uparrow$' represent exponentiation, $a \uparrow n = a^n$):

$$[a\ +]\ \$\ \sigma = \sigma$$

$$[a\ \times]\ \$\ \sigma = [a\ +]$$

$$[a\ \uparrow]\ \$\ \sigma = [a\ \times]$$

$$[a\ -]\ \$\ \sigma = \sigma^{-1}$$

$$[-\ a]\ \$\ \sigma = \sigma$$

The first of these equations follows from Corollary 6-1 and the observation that $[a\ +] = \sigma^a$. The next two equations follow from the definition and theorem below, which generalizes Corollary 6-1.

Definition 4: We write '$(\text{power}_a\ f)\ n$' for the n^{th} power of function f applied to initial value a: $(\text{power}_a\ f)\ n = f^n\ a$. This is defined recursively:

$$(\text{power}_a\ f)\ 0 = a$$

$$(\text{power}_a\ f)\ (n+1) = f\ [(\text{power}_a\ f)\ n], \text{ for } n \geqslant 0$$

We call $\text{power}_a\ f$ 'the power from a of f'.

Theorem 11: The difference of the power (from a) of f with respect to successor is f:

$$(\text{power}_a\ f)\ \$\ \sigma = f$$

Proof: This follows directly from the definition of 'power':

$$(\text{power}_a\ f)\ (\sigma\ n) = f\ [(\text{power}_a\ f)\ n]$$

Therefore, $(\text{power}_a\ f) \circ \sigma = f \circ (\text{power}_a\ f)$. □

Now observe that

$$\lfloor a\ + \rfloor = \mathrm{power}_a\ \sigma$$

$$\lfloor a\ \times \rfloor = \mathrm{power}_0\ \lfloor a\ + \rfloor$$

$$\lfloor a\ \uparrow \rfloor = \mathrm{power}_1\ \lfloor a\ \times \rfloor$$

$$\lfloor a\ - \rfloor = \mathrm{power}_a\ \sigma^{-1}$$

The differences of these functions then follow from the theorem.

6. Recursive Definitions

Consider the following equations, which define the length function on LISP-like lists:

$$\text{length nil} = 0$$

$$\text{length } (x\ \&\ y) = 1 + \text{length } y$$

(Here 'x & y' denotes the result of prefixing x on the list y – the LISP 'cons' operation.) The second equation is a functional difference equation, as can be seen by writing it in the form:

$$\text{length} \circ \lfloor x\ \& \rfloor = \sigma \circ \text{length}$$

Hence it is easy to see that the change in length with respect to prefixing is the successor:

$$\text{length}\ \$\ \lfloor x\ \& \rfloor = \sigma.$$

On the other hand, if we were to define length recursively, we would write something like this:

$$\text{length } L = \begin{cases} 0, & \text{if } L = \text{nil} \\ 1 + \text{length (rest } L), & \text{if } L \neq \text{nil} \end{cases}$$

This corresponds to the equations

$$\text{length nil} = 0$$

$$\text{length } L = 1 + \text{length (rest } L), \text{ for } L \neq \text{nil}$$

The second equation here is also a sort of difference equation, but it does not fit our earlier form. Written in terms of compositions it is:

$$\text{length} \circ \mathbf{I}_N = \sigma \circ \text{length} \circ \text{rest}$$

where we have composed length with $\mathbf{I}_N$ to restrict its domain to nonnull lists (taking N to be the set of nonnull lists).

What is the relationship between the two difference equations satisfied by length? Consider the first difference equation:

$$\text{length} \circ [x\ \&] = \sigma \circ \text{length}$$

Compose with the inverse of $[x\ \&]$ on both sides:

$$\text{length} \circ [x\ \&] \circ [x\ \&]^{-1} = \sigma \circ \text{length} \circ [x\ \&]^{-1}$$

Now $[x\ \&] \circ [x\ \&]^{-1}$ is the identity restricted to the range of $[x\ \&]$, which in turn is the set of lists beginning with x. Hence, if N_x is the set of nonnull lists beginning with x, then

$$\text{length} \circ \mathbf{I}_{N_x} = \sigma \circ \text{length} \circ [x\ \&]^{-1}$$

This looks almost like our second difference equation:

$$\text{length} \circ \mathbf{I}_N = \sigma \circ \text{length} \circ \text{rest}$$

We can see their relationship as follows.

The meaning of $[x\ \&]$ is to put x on the front of its argument. Hence, the meaning of $[x\ \&]^{-1}$ is to take x off the front of its argument. On the other hand 'rest' takes the first thing off the front of its argument no matter what it is. Hence $[x\ \&]^{-1}$ is like 'rest' except that it's defined only on lists whose first element is x. That is, $[x\ \&]^{-1}$ is a proper subfunction of 'rest', $[x\ \&]^{-1} \subset \text{rest}$.

Now we make two simple observations. First, the set of all nonnull lists is the union, for all x, of the nonnull lists that begin with x:

$$N = \bigcup_x N_x$$

Second, the 'rest' function, which deletes the first element of a list no matter what it is, is the union of all the functions $[x\ \&]^{-1}$, which delete x from the front of a list:

$$\text{rest} = \bigcup_x [x\ \&]^{-1}$$

It is now easy to show the two difference equations are equivalent.

Theorem 12: Suppose that

$$\text{length} \circ \mathbf{I}_{N_x} = \sigma \circ \text{length} \circ [x\ \&]^{-1}$$

is true for all x. Then

$$\text{length} \circ \mathbf{I}_N = \sigma \circ \text{length} \circ \text{rest}$$

The converse also holds.

Proof: To prove the first implies the second we have:

$$\begin{aligned} \text{length} \circ \mathbf{I}_N &= \text{length} \circ \bigcup_x \mathbf{I}_{N_x} \\ &= \bigcup_x (\text{length} \circ \mathbf{I}_{N_x}) \\ &= \bigcup_x (\sigma \circ \text{length} \circ [x\ \&]^{-1}) \\ &= \sigma \circ \text{length} \circ \bigcup_x [x\ \&]^{-1} \\ &= \sigma \circ \text{length} \circ \text{rest} \end{aligned}$$

To prove the second implies the first we restrict both sides to N_x:

$$\text{length} \circ \mathbf{I}_N \circ \mathbf{I}_{N_x} = \sigma \circ \text{length} \circ \text{rest} \circ \mathbf{I}_{N_x}$$

Observing that $\text{rest} \circ \mathbf{I}_{N_x} = [x\ \&]^{-1}$:

$$\text{length} \circ \mathbf{I}_{N_x} = \sigma \circ \text{length} \circ [x\ \&]^{-1}$$

□

7. Definition of Integral

In this section we consider a *functional integration* operation that is inverse to the functional difference. That is, given isomorphic functions g and h we want to find an f that satisfies the functional difference equation

$$g = f \$ h$$

Since this equation states that g is the isomorphic image under f of h, our goal can be viewed as finding an isomorphism between g and h.

In general there may be many isomorphisms between two relations. Since this implies that the solution to a functional difference equation is often *underdetermined* by the equation, to determine a particular

solution it's necessary to specify a *boundary function* b contained in the solution. Thus the solution to the equation is required to be an extension of the boundary function (i.e., $b \subseteq f$).

As we did for functional differences, here also we want to permit the case where g is isomorphic to a part of f; that is, the case in which f is not defined for some members of h. This leads us to:

Definition 5: Let arbitrary relations g, h and b be given. If there is a minimum isomorphism f, with $b \subseteq f$, satisfying the equation

$$f \circ h = g \circ f$$

then we call f the *definite functional integral* with respect to h, from b, of g. We write f:

$$h \; \Phi_b \; g$$

This is read "the h integral from b of g."

Theorem 13: If the definite functional integral exists, then it satisfies the equation:

$$\left(h \; \Phi_b \; g \right) \circ h = g \circ \left(h \; \Phi_b \; g \right)$$

Proof: Follows immediately from definition. □

Next we explore the conditions under which functional integrals exist.

Lemma 13-1: R and S are isomorphic relations if and only if there exists a one-to-one function ϕ such that $S = \phi \; \$ \; R$ and mem R = dom ϕ and mem S = rng ϕ.

Proof: This follows easily from the definition of isomorphic relations in [Carnap58]. □

The preceding lemma says that two relations are isomorphic if there is a one-to-one function (isomorphism) between them that preserves all the structure of both. Following [Carnap58] we call such a function a *correlator* between the relations. In the following definition we give a name to the case in which the correlator does not preserve all the structure of one of the relations.

Theorem 14: If g and h are isomorphic relations, and b is a subset of exactly one isomorphism between g and h, then $h \; \Phi_b \; g$ exists.

Proof: We know (by hypothesis) that there is a unique f such that $b \subseteq f$ and f is an isomorphism between g and h. Note that since f is a correlator between g and h, $g = f \,\$\, h$ and mem h = dom f. Hence we can apply Cor. 4.1 and conclude that $f \,\$\, h$ satisfies the difference equation $fh = gf$. This f is minimal, since eliminating any element of f would result in f not being defined for all members of h (and hence not an isomorphism between g and h). Hence f is the functional integral $h \, \Phi_b \, g$. □

8. Examples of Integrals

We can perform a number of functional integrations based an previously established functional differences. First note

Theorem 15: The power operator satisfies the difference equation:

$$(\text{power}_a\, f) \circ \sigma = f \circ (\text{power}_a\, f)$$

Proof: We have already shown (Theorem 11) that $(\text{power}_a\, f) \,\$\, \sigma = f$, so it remains to show that the difference equation has a solution. But we know a solution exists if and only if

$$\text{dom}\,(\text{power}_a\, f)\sigma \subseteq \text{dom}\,(\text{power}_a\, f)$$

But since dom σ = dom $(\text{power}_a\, f)$ the above condition holds. □

Corollary 15-1: The σ integral from $(0,\, a)$ of f is the power from a of f:

$$\sigma \, \Phi_{(0,\, a)} \, f = \text{power}_a\, f$$

Proof: Since $(\text{power}_a\, f)\, 0 = a$ we know that the boundary relation $(0,\, a) \subseteq \text{power}_a\, f$. The result follows because, by the preceding theorem, the power operator satisfies the difference equation (and it is clear that it's the only isomorphism to do so). □

Corollary 15-2: We have the following functional integrals:

$$\begin{aligned}
\sigma \, \Phi_{(0,\, a)} \, \sigma &= [a\ +] \\
\sigma \, \Phi_{(0,\, 0)} \, [a\ +] &= [a\ \times] \\
\sigma \, \Phi_{(0,\, 1)} \, [a\ \times] &= [a\ \uparrow]
\end{aligned}$$

$$\sigma\, \Phi_{(0,\,a)}\, \sigma^{-1} \quad = \quad [a\ -]$$

$$\sigma\, \Phi_{(0,\,-a)}\, \sigma \quad = \quad [-\ a]$$

Proof: Immediate application of preceding corollary and definitions of the integrands. □

9. Computing Integrals of Sequences

Consider the difference equation $f \circ h = g \circ f$ and suppose that we are given g and h and want to determine f. We begin by investigating a restricted class of equations: those in which both g and h are well-founded sequences. So further suppose that h is the sequence $(x_0, x_1, \cdots)$ and g is the sequence $(y_0, y_1, \cdots)$. Thus $x_i = h^i x_0$ and $y_i = g^i y_0$. Finally suppose that the solution is required to extend the boundary function $b = (x_0, y_0)$ that maps x_0 into y_0.[2] This situation can be diagramed:

$$\begin{tikzcd}
x_0 \arrow[r, "h"] \arrow[d, "b"'] & x_1 \arrow[r, "h"] & x_2 \arrow[r, "h"] & \cdots \\
y_0 \arrow[r, "g"'] & y_1 \arrow[r, "g"'] & y_2 \arrow[r, "g"'] & \cdots
\end{tikzcd}$$

Our goal is to find an isomorphism f connecting each x_i with the corresponding y_i, that is, $y_i = fx_i$. We will construct f inductively.

Referring to the above diagram it can be seen that we can get from x_0 to y_0 by following edge b. We can get from x_1 to y_1 by following h^{-1} (i.e., h backward), then b, then h. Similarly, to get from x_2 to y_2, we follow h^{-1} twice, then b, then h twice. Thus f can be expressed as the infinite union of the paths (x_i, y_i), which can be expressed in terms of h^{-1}, b and h:

$$\begin{aligned} f &= (x_0, y_0) \cup (x_1, y_1) \cup (x_2, y_2) \cup (x_3, y_3) \cup \cdots \\ &= b \cup (g \circ b \circ h^{-1}) \cup (g^2 \circ b \circ h^{-2}) \cup (g^3 \circ h^{-3}) \cup \cdots \end{aligned}$$

Hence,

$$f = b \cup gbh^{-1} \cup g^2bh^{-2} \cup g^3bh^{-3} \cup \cdots = b \cup \bigcup_{i=1}^{\infty} g^i bh^{-i}$$

2. Actually, in this case there is only one possible isomorphism between the relations g and h, so the specification of the boundary function is redundant. However, we are trying to develop a general method.

We would like a finite representation for f. Therefore compose on the left with g and the right with h^{-1} to get:

$$gfh^{-1} = gbh^{-1} \cup g^2bh^{-2} \cup g^3bh^{-3} \cup \cdots = \bigcup_{i=1}^{\infty} g^i bh^{-i}$$

This is the original equation except for the b term, so it is easy to see

$$f = b \cup gfh^{-1}$$

Thus we have a recursive definition of f, the solution to the functional difference equation. Our next goal will be to obtain a finite, nonrecursive expression for the solution.

Define the functional product $f \,||\, g$ between two functions:

$$(f \,||\, g) <x, y> = <f\ x, g\ y>$$

Thus, if f: $D \rightarrow R$ and g: $D' \rightarrow R'$, then

$$(f \,||\, g): (D \times D') \rightarrow (R \times R')$$

Now it is easy to see that

$$\begin{aligned}
<x_1, y_1> &= (h \,||\, g) <x_0, y_0> \\
<x_2, y_2> &= (h \,||\, g) <x_1, y_1> = (h \,||\, g)^2 <x_0, y_0> \\
<x_3, y_3> &= (h \,||\, g) <x_2, y_2> = (h \,||\, g)^3 <x_0, y_0> \\
&\vdots \\
<x_i, y_i> &= (h \,||\, g)^i <x_0, y_0>
\end{aligned}$$

Hence,

$$f = \{ <x_0, y_0>, (h \,||\, g) <x_0, y_0>, (h \,||\, g)^2 <x_0, y_0>, (h \,||\, g)^3 <x_0, y_0>, \cdots \}$$

It can be seen that f results from applying all the functions $(h \,||\, g)^0$, $(h \,||\, g)^1$, $(h \,||\, g)^2$, $\cdots$ to the initial pair $<x_0, y_0>$. We will show that this is the *image* of the relation (x_0, y_0) under the transitive closure:

$$(h \,||\, g)^* = (h \,||\, g)^0 \cup (h \,||\, g)^1 \cup (h \,||\, g)^2 \cup \cdots$$

Therefore define the image function:

$$\text{img}\ f\ S = \{ y \mid \exists x \in S \,[<x, y> \in f] \}$$

We begin by proving

Lemma 15-1: For any relations f and g and any set S,

$$\text{img } f\ S \cup \text{img } g\ S = \text{img } (f \cup g)\ S.$$

Proof:

$$\begin{aligned}
y \in \text{img } (f \cup g)\ S &\iff \exists\, x \in S\ [<x, y> \in f \cup g] \\
&\iff \exists\, x \in S\ [<x, y> \in f \vee <x, y> \in g] \\
&\iff \exists\, x \in S\ [<x, y> \in f] \vee \exists\, x \in S\ [<x, y> \in g] \\
&\iff y \in \text{img } f\ S \vee y \in \text{img } g\ S \\
&\iff y \in (\text{img } f\ S \cup \text{img } g\ S)
\end{aligned}$$

□

Note that this lemma applies when f and g are functions, although in that case $f \cup g$ will not be a function unless f and g have disjoint domains.

We already know that

$$f = (x_0, y_0) \cup (x_1, y_1) \cup (x_2, y_2) \cup \cdots$$

Now observe that

$$\begin{aligned}
(x_i, y_i) &= \{<x_i, y_i>\} \\
&= \{(h \mid\mid g)^i <x_0, y_0>\} \\
&= \text{img } (h \mid\mid g)^i \{<x_0, y_0>\} \\
&= \text{img } (h \mid\mid g)^i (x_0, y_0)
\end{aligned}$$

Hence, taking $b = (x_0, y_0)$,

$$f = [\text{img } (h \mid\mid g)^0\ b] \cup [\text{img } (h \mid\mid g)^1\ b] \cup [\text{img } (h \mid\mid g)^2\ b] \cup \cdots = \bigcup_{i=0}^{\infty} \left[\text{img } (h \mid\mid g)^i\ b \right]$$

which by the lemma is[3]

$$f = \text{img } [(h \mid\mid g)^0 \cup (h \mid\mid g)^1 \cup (h \mid\mid g)^2 \cup \cdots]\ b = \text{img} \left[\bigcup_{i=0}^{\infty} (h \mid\mid g)^i \right] b$$

3. Actually, by a transfinite extension of the lemma.

But the expression in brackets is just the transitive closure of $(h \parallel g)$, so we have

$$f = \text{img } (h \parallel g)^* b$$

as a solution to the functional difference equation

$$f \circ h = g \circ f$$

This result is summarized in

Theorem 16: If h and g are well-founded sequences and $b = (x_0, y_0)$, where x_0 and y_0 are the initial members of h and g, respectively, then the *definite functional integral* with respect to h of g, generated from boundary isomorphism b, is

$$h \, \Phi_b \, g = \text{img } (h \parallel g)^* \, b$$

Proof: The proof is clear from the derivation. □

Therefore we introduce

Definition 6: If g and h are well-founded sequences then the *indefinite functional integral* with respect to h of g is:

$$h \, \Phi \, g = \text{img } (h \parallel g)^*$$

This is read "the h integral of g."

Our next task must be to prove that this solution satisfies the difference equation.

Theorem 17: If g and h are well-founded sequences, then the definite functional integral satisfies the equation

$$\left(h \, \Phi_b \, g \right) \circ h = g \circ \left(h \, \Phi_b \, g \right)$$

Proof: As shown in the proof of Lemma 15-1,

$$h \, \Phi_b \, g = (x_0, y_0) \cup (x_1, y_1) \cup (x_2, y_2) \cup \cdots$$

Hence the left-hand side of the difference equation is:

$$(h \, \Phi_b \, g) \circ h = [(x_0, y_0) \cup (x_1, y_1) \cup (x_2, y_2) \cup \cdots] \circ h$$

$$= (x_0, y_1) \cup (x_1, y_2) \cup (x_2, y_3) \cup \cdots$$

The last equation follows from $x_{i+1} = hx_i$. Now we turn to the right-hand side of the equation:

$$g \circ (h \,\Phi_b\, g) = g \circ [(x_0, y_0) \cup (x_1, y_1) \cup (x_2, y_2) \cup \cdots]$$

$$= (x_0, y_1) \cup (x_1, y_2) \cup (x_2, y_3) \cup \cdots$$

The last equation follows from $y_{i+1} = gy_i$. The identity of these results proves the theorem. □

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11. References

[Carnap58] Carnap, Rudolf, *Introduction to Symbolic Logic and its Applications,* Dover, New York: 1958.

[MacLennan83] MacLennan, Bruce J., Relational Programming, Naval Postgraduate School Computer Science Department Technical Report NPS52-83-012.

[Paige&Koenig82] Paige, Robert, and Koenig, Shaye, Finite differencing of computable expressions, *ACM Trans. Prog. Lang. and Sys. 4,* 3 (July 1982), 402-454.

[Wile73] Wile, D. S., A generative nested-sequential basis for general purpose programming languages, Ph.D. dissertation, Dept. of Computer Science, Carnegie-Mellon Univ., Pittsburgh: 1973.

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